Wireless Receivers [Lecture 1] 18/9/2020

Discrete time Model + Modulation

Objective Get a bit stream $b_0, b_1, b_2, \ldots$ from A to B through some physical medium (analog, cont. time)

discrete time bits 0/1 → cont.-time analog ~ → disc. time digital 0/1

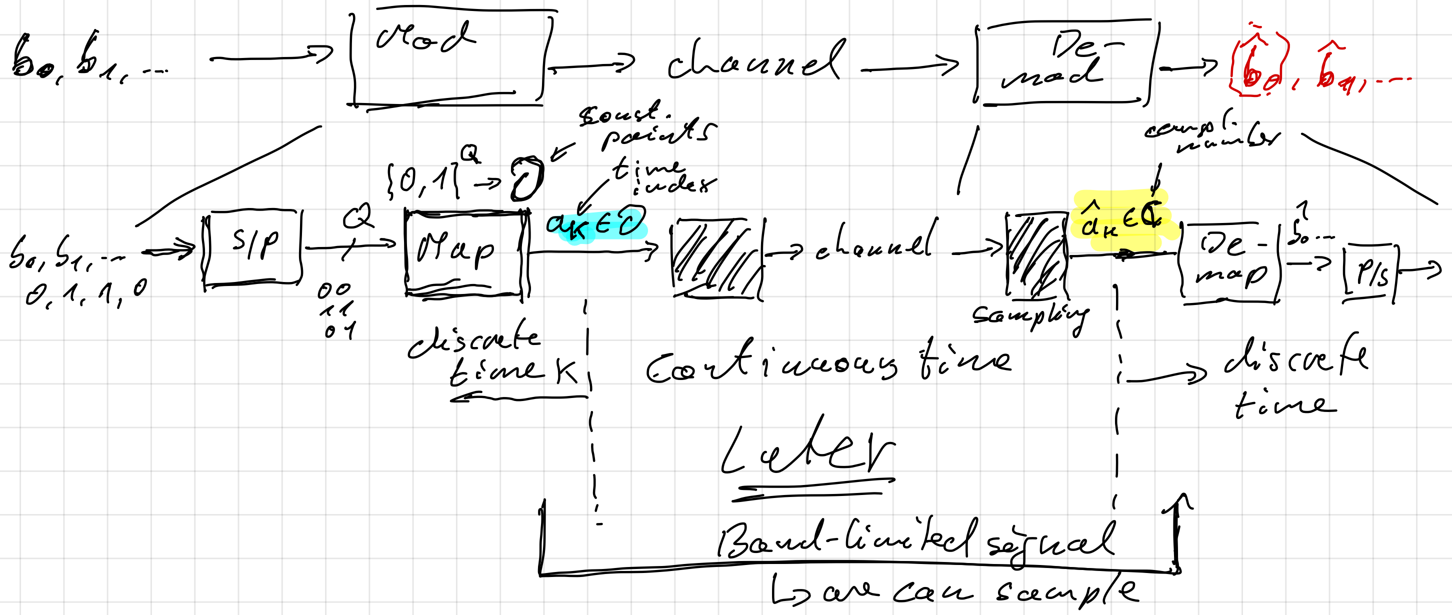
Bits are not physical quantities

Modulation { ① Map. disc. time stream of bits to analog. cont. time signal

② send signal

De-mod. { ③ Recover bits

$\hat{a}$ is an estimate of a $\quad \hat{a} \neq a$



Map: Bits → Symbols

Demap: Complex numbers → Symbols → bits

① Mapper: Map Q bits into one Symbol

Each symbol contains Q bits

↳ „Q bits per channel use"

Qbpcu Spectral efficiency

$\{0,1\}^Q \rightarrow \mathcal{O}$ = Set of Const. Points

$\{x_1, x_2, ... x_M\} \in \mathbb{C}$ with $M = 2^Q$

② Channel: $a_K \in \mathcal{O} \longrightarrow \hat{a}_K \in \mathbb{C}$

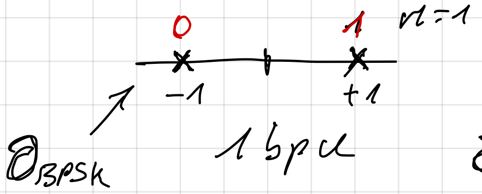
③ Demapper: - ideally $\hat{a}_K \in \mathcal{O} \rightarrow \{0,1\}^Q$

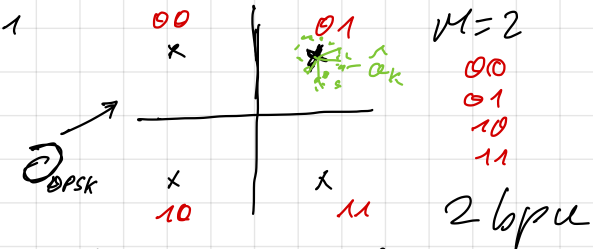
- reality $\hat{a}_K \in \mathbb{C}$ due to noise

The set of „constellation points" $\mathcal{O}$

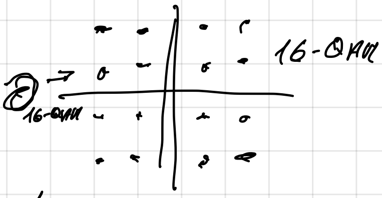
$\mathcal{O} = \{x_1, x_2, ... x_M\}$ where x_n could be complex or real

Const. Point Sets : Each const. point needs a label ← binary





- regular
- equally spaced
- unique



QAM: $M = 2^Q$ (M-QAM) $Q = 2,4,6,8,...$

- Compl. valued const. points
- useful: Compl. numbers ~~ca~~ can represent phase and magnitude of RF signals

QAM alphabets constructed by placing points on a grid

$\Re\{x_i\} \in \{\pm 1, \pm 3, \pm 5, ...\}$ $\Im\{x_i\} = \{\pm 1, \pm 3, \pm 5, ...\}$

$\underline{Q=4}$ $\Re\{x_i\} = \underbrace{\{\pm 1, \pm 3\}}_{\times 4}$ same for Img. $\underset{\times 4}{\rightarrow}$ $\underline{16}$

Signal Energy of (un-normalized) QAM:

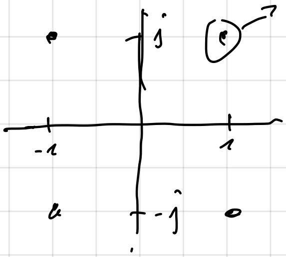
- Signal Energy of each symbol $a_K \in \mathcal{O}$: $P_K = |a_K|^2$
  is different for each sent symbol

- Average Signal Energy $\bar{P} = \mathcal{E}_k\{|a_k|^2\}$

Assume: all symbols equally likely $P_r(x_i) = 1/M$

$$\Rightarrow \bar{P} = \sum_{i=1}^{M} |x_i|^2 \underbrace{P_r(x_i)}_{1/M}$$

Example: QPSK $(M=4)$



$$\left.\begin{array}{l} 1+j \\ 1-j \\ -1+j \\ -1-j \end{array}\right\} \bar{P} = 2$$

16-QAM $(M=16)$ $\bar{P} = 10$

$$\bar{P}_{M\text{-}QAM} = \frac{2}{3}(M-1)$$

The AWGN: Additive White Gaussian Noise

- Simple but freq. used
- Impact of thermal noise

$$\begin{tikzcd} \underset{\text{from Tx}}{a_k^{\in \mathcal{O}}} \arrow[r] & \oplus \arrow[r] & r_k = \overset{\in \mathcal{O}}{a_k} + \overset{\mathcal{CN}}{w_k} \notin \mathcal{O} \\ & w_k \arrow[u] & \end{tikzcd}$$

Noise characterized by:

- Noise power $\sim \mathcal{CN}(0, \underbrace{\sigma_w^2}_{\text{Noise power}})$
- PDF ← Gauss

Characterize signal quality @ Rx

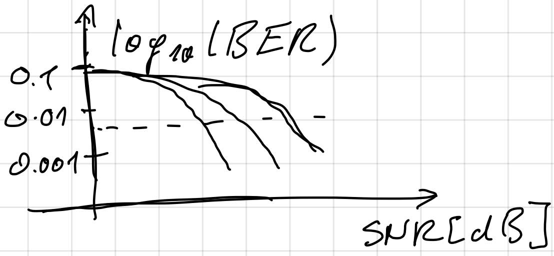
Signal-to-noise ratio $\text{SNR} = \frac{\bar{P}}{P_N} = \frac{\overset{\bar{P}}{\mathcal{E}\{|a_k|^2\}}}{\underbrace{\mathcal{E}\{|w_k|^2\}}_{\sigma_w^2}}$

Usually measure in [dB]:

$$\text{SNR[dB]} = 10 \cdot \log\left(\frac{\bar{P}}{P_N}\right) \text{[dB]}$$

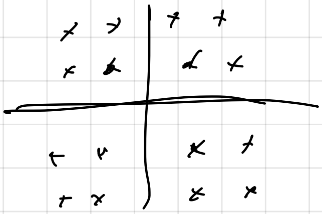
How to compare Comm. Systems

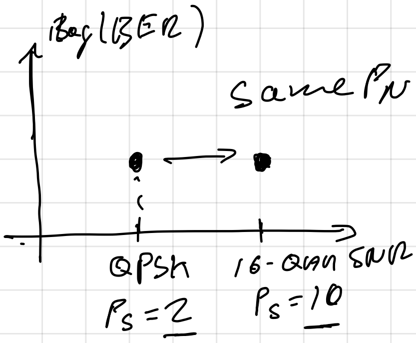
- Quality metric: ERROR-RATE $(\log_{10})$ vs. SNR [dB]
- SNR is fct. of Tx power



Example: QPSK vs. 16 QAM (un-normalized)

$\bar{P}(QPSK) < \bar{P}(16\text{-}QAM)$





→ pay attention to TX power

Better: always normalize TX-power, ideall $\bar{P} = 1$

$$\Rightarrow SNR = \frac{1}{P_N} \quad \left(\frac{P_s}{P_N}\right)$$

Normalized Constellations

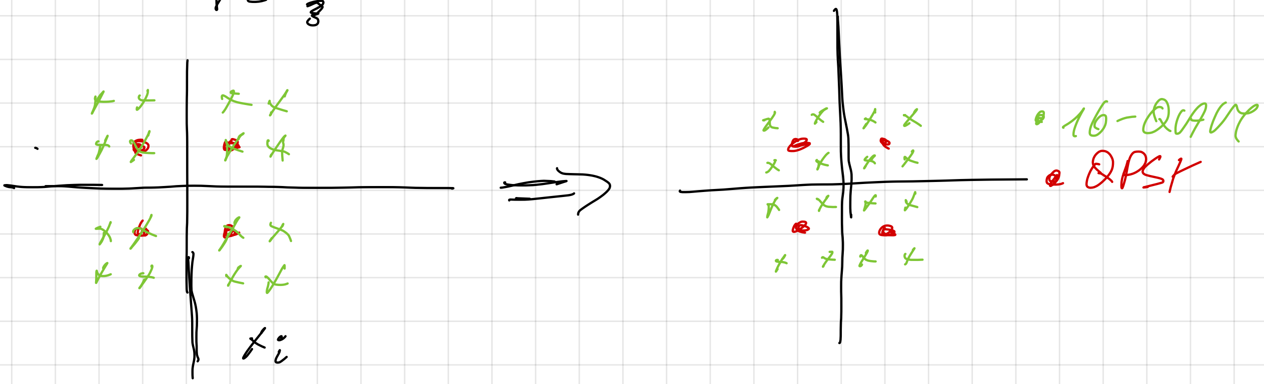
x_i : Const. w/o normalization $\quad \bar{P} = P_{M\text{-}QAM} \quad SNR = \frac{\bar{P}}{P_N}$

↓ normalize

$\tilde{x}_i$: with $\bar{P} = 1 \quad SNR = \frac{1}{P_N}$

$$\tilde{x}_i = \frac{1}{\sqrt{P_{M\text{-}QAM}}} x_i \rightarrow \bar{P} = E\{ \frac{1}{P_{M\text{-}QAM}} |x_i|^2 \} = 1$$

$$= x_i = \frac{1}{\sqrt{2\frac{M-1}{3}}}$$

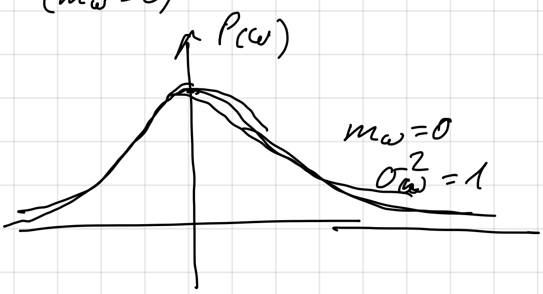


Back to the Noise

Assumptions: Gauss + zero mean ($m_\omega = 0$)

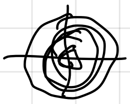
Gaussian noise PDF (real)

$$P(\omega) = \frac{1}{\sqrt{2\pi\sigma_\omega^2}} e^{-\frac{|\omega - m_\omega|^2}{2\sigma_\omega^2}}, \omega \in \mathbb{R}$$



Noise Power: $P_N = \sigma_\omega^2$

Complex Gaussian noise (circular symmetric)



$\omega = \omega_R + j\omega_I$

$\omega \sim \mathcal{CN}(0, \underset{\uparrow}{\sigma_\omega^2}) \rightarrow \mathrm{Re}\{\omega\}: \mathcal{N}(0, ?)$

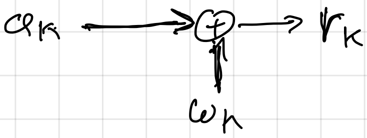
$\mathrm{Im}\{\omega\}: \mathcal{N}(0, ?)$

$$\sigma_\omega^2 = \mathcal{E}\{|\omega|^2\} = \mathcal{E}\{|\omega_R|^2\} + \mathcal{E}\{|\omega_I|^2\} = \sigma_{\omega_R}^2 + \sigma_{\omega_I}^2$$

$$\Rightarrow \underline{\omega_R}, \underline{\omega_I}: \underline{\mathcal{N}\left(0, \frac{\sigma_\omega^2}{2}\right)}$$

Discrete channel output: (input of demod)

- Two prob. quantities: noise (ω_k) and symbol (a_k)



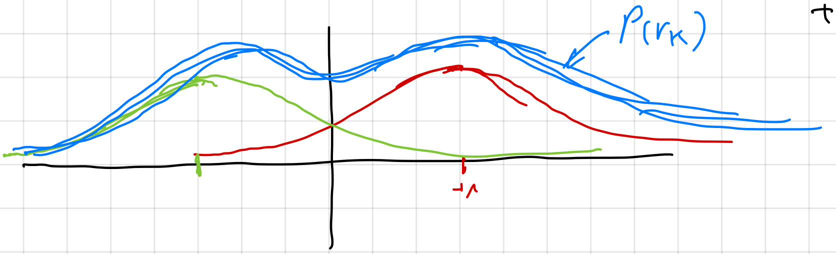
$r_k = \underline{a_k} + \omega_k$

- Consider the PDF of r_k $\Pr(r_k)$

Example of BPSK $\Pr(a_k = \pm 1) = 1/2$

$$\Pr(r_k) = \boxed{P(a_k = +1)\, \underline{P(r_k | a_k = 1)}} + \boxed{P(a_k = -1)\, \underline{\Pr(r_k | a_k = -1)}}$$

↑ Gauss PDF with mean of a_k



Receiver: based on r_K decide on if $\hat{a}_K = x_i$

$$\boxed{\text{The ML-principle: minimize probab. of error}}$$

Idea: Choose the const. point x_i that maximizes $Pr(a_K = x_i | r_K)$

what we receive

$$\hat{a}_K = \arg\max(P(a_K = x_i | r_K))$$

?

Difficult to obtain →

But we do know $Pr(r_K | a_K = x_i)$

How to rewrite $Pr(a_K = x_i | r_K)$ in terms of $Pr(r_K | a_K = x_i)$

- Bayes Rule: $P(a_K | r_K) = \frac{P(r_K | a_K) P(a_K)}{P(r_K)}$

assume $P(a_K) = \frac{1}{M}$ all sym. equally likely

$P(r_K)$: indep. $a_K \Rightarrow$ no influence

ML detection

$$\hat{a}_K = \arg\max_{x_i \in \Omega} P(r_K | a_K = x_i)$$

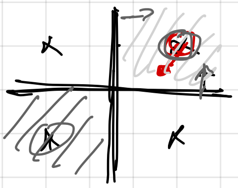
ML estimation for AWGN

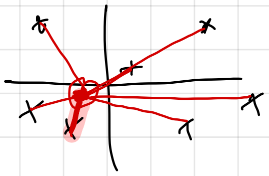
$$Pr(r_K | a_K = x_i) = \frac{1}{2\pi\sigma_n^2} e^{-\frac{|r_K - x_i|^2}{2\sigma^2}}$$

Maximizing $P(r_K | a_K = x_i)$ equivalent to minimizing $|r_K - x_i|^2$

ML for AWGN

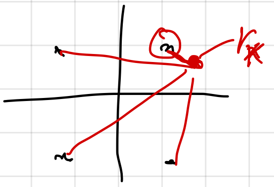
$$\Rightarrow \hat{a}_K = \arg\min_{x_i} |r_K - x_i|^2$$





Error Probability

- ML detection: success is not guaranteed
- Error: r_k lies closer to a point x_j that is different from the point we sent x_i

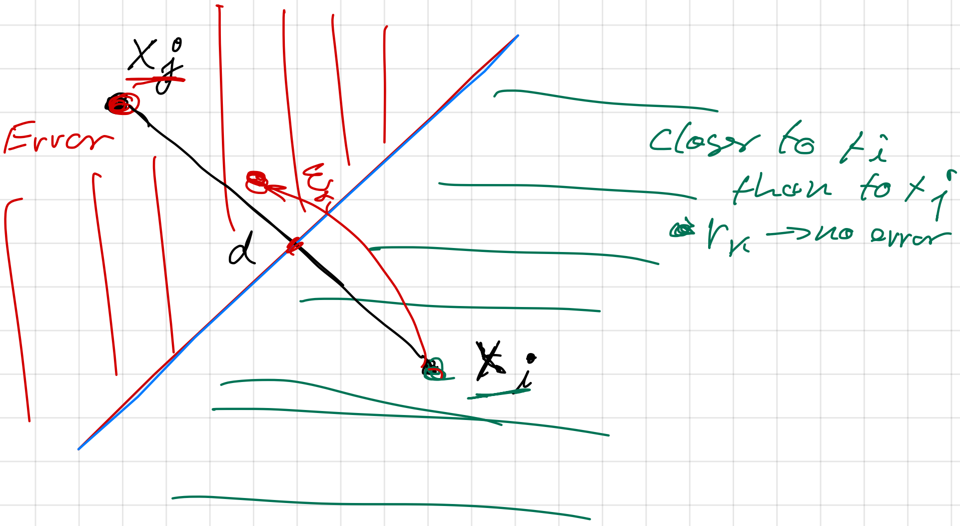


Error Sent: x_i

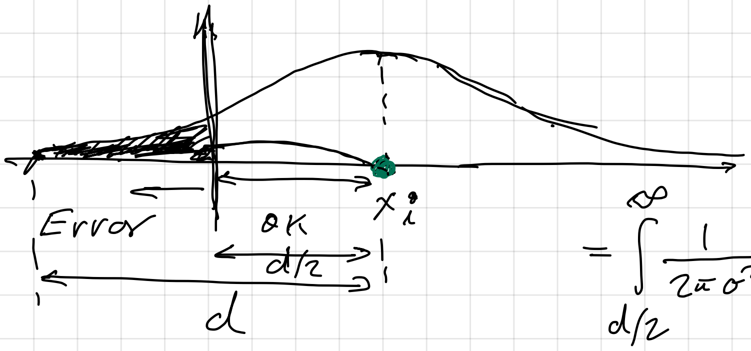
make a choice for x_j $j \neq i$

$$|r_k - x_j|^2 < |r_k - x_i|^2$$

Example:



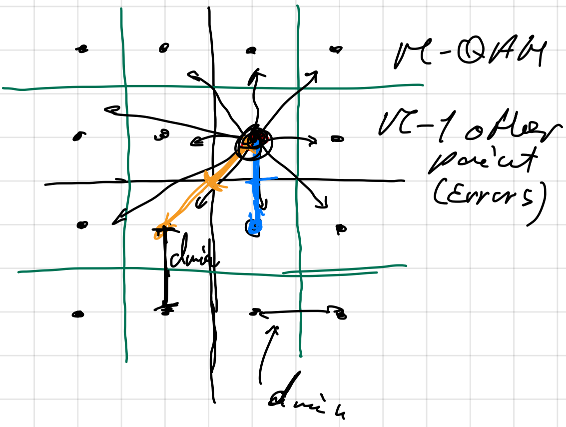
1D:



$Pr(w_k > \frac{d}{2})$: Error

$$= \int_{d/2}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{|w|^2}{2\sigma^2}} = Q\left(\sqrt{\frac{d^2}{4} \underbrace{\frac{1}{\sigma^2}}_{SNR}}\right)$$

QAM:
- Can make a mistake to any other point
- Decision boundaries are different



Approx: Consider just d_{min}

$P(x_j, x_i)$: PEP $\leq Q\left(\sqrt{\frac{d_{min}^2}{4} SNR}\right)$

M-1 : Error events

$P_e \leq (M-1)\, Q\left(\sqrt{\frac{d_{min}^2}{4} SNR}\right)$ ~ Bounds